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On the existence of slow manifolds for problems with different timescales

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We consider time dependent systems of partial differential equations (PDE) whose solutions can vary on two different timescales. An example is given by the Navier-Stokes equations for slightly compressible flows. By proper initialization, the fast timescale can be suppressed to any given order; however, this does generally not imply the existence of a slow manifold. Since the PDE solutions are uniformly smooth in space, one can approximate the PDE system by a finite dimensional Galerkin system. Under suitable assumptions, this finite dimensional dynamical system will have a slow manifold.

1. Introduction

In applications, the initial value problem for systems of partial differential equations which allow solutions on different timescales typically has the form

$$\frac{\partial u}{\partial t} = \epsilon^{-1} P_0\left(\frac{\partial}{\partial x}\right) u + P_1\left(u, \frac{\partial}{\partial x}\right) u + \nu P_2\left(\frac{\partial}{\partial x}\right) u + F(x,t), \quad t \geqslant 0, \tag{1.1}$$

$$u(x,0) = f(x), \quad x = (x_1, \ldots, x_s) \in \mathbb{R}^s.$$

Here $u = (u^{(1)}, \ldots, u^{(n)})'$ is a real vector function with n components. The operators P_0 and P_1 are first-order differential operators of the form

$$P_0\left(\frac{\partial}{\partial x}\right) = \sum_{j=1}^{s} A_j \frac{\partial}{\partial x_j}, \quad A_j = A_j^* \in \mathbb{R}^{n\times n},$$

$$P_1\left(u, \frac{\partial}{\partial x}\right) = \sum_{j=1}^{s} B_j(u) \frac{\partial}{\partial x_j}, \quad B_j = B_j^* \in \mathbb{R}^{n\times n},$$

i.e. the coefficients are real $n \times n$ hermitean matrices. The A_j are constant matrices and the B_j are polynomials in the components of u. The operator $P_2(\partial/\partial x)$ is a second-order differential operator with constant real coefficients. In applications, νP_2 represents the dissipation present in the system. The parameters $\epsilon > 0$ and $\nu > 0$ are small constants which measure the difference in timescales and the level of dissipation, respectively. (The coefficients of P_0, P_1, P_2 have been normalized to be of order $O(1)$.)

We are interested in solutions which are 2π-periodic in all space variables x_j. We assume that the initial data f and the forcing function F are C^∞-smooth and 2π-periodic in all space variables x_j. For later purposes, we also assume that F is defined for $t \geqslant -1$.

An example of a problem with different timescales is given by low-Mach-number flow. In two space dimensions, it is described by the Navier–Stokes equations, which in slightly simplified form are given by

$$\left.\begin{aligned} &u_t+uu_x+vu_y+p_x=\nu\Delta u, \quad \Delta u=u_{xx}+u_{yy},\\ &v_t+uv_x+vv_y+p_y=\nu\Delta v,\\ &\epsilon^2(p_t+up_x+vp_y)+(u_x+v_y)=0. \end{aligned}\right\} \tag{1.2}$$

Here u and v denote the velocity components in the x and y-directions, respectively, and p represents the pressure. The problem (1.2) has the form (1.1) if we symmetrize the system by introducing $\epsilon p=\tilde{p}$ as new variable. A discussion of the system from the point of view of different timescales has recently been presented by Kreiss *et al.* (1991). See also Klainerman & Majda (1982).

In many applications, one is not interested in the part of the solution which varies on the fast timescale. Therefore, one wants to determine initial data for which the fast timescale is not activated. In §2 we will consider systems with constant coefficients and show how we can achieve this goal under suitable assumptions. In §3 we will generalize the results to nonlinear equations. Finally, in §4 we will approximate (1.1) by a system of ordinary differential equations and give conditions for the existence of a slow manifold of the approximating system.

We remark that the construction of a slow manifold for a highly oscillatory problem with different timescales is usually more delicate than the construction of a so-called inertial manifold or approximate inertial manifold for a dissipative problem. Inertial manifolds are exponentially attracting; if initial data are chosen off the manifold, then the trajectory shows transient behaviour, converging to the inertial manifold at an exponential rate. For highly oscillatory problems, initial data off the slow manifold lead to highly oscillatory behaviour, which may or may not die out. In this case, the interaction of the slow and the fast scale is of interest; see Kreiss & Lorenz (1994) for an estimate of the interaction.

2. Systems with constant coefficients

In this section we assume that the coefficients B_j of the operator P_1 are constant matrices. In this case, we can use Fourier expansion to transform the system (1.1) and obtain

$$\hat{u}_t(\omega,t)=|\omega|\,\hat{P}(\omega)\,\hat{u}(\omega,t)+\hat{F}(\omega,t), \quad \hat{u}(\omega,0)=\hat{f}(\omega). \tag{2.1}$$

Here $\omega=(\omega_1,\ldots,\omega_s)\in\mathbb{Z}^s$ denotes the (real) dual variable to the space variable $x\in\mathbb{R}^s$ and

$$\hat{P}(\omega)=\epsilon^{-1}P_0(\mathrm{i}\omega')+P_1(\mathrm{i}\omega')+\nu\,|\omega|\,P_2(\mathrm{i}\omega'), \tag{2.2}$$

where $\omega'=\omega/|\omega|$, $|\omega|^2=\Sigma\omega_j^2$. (For simplicity of presentation, we will always assume that $\hat{f}=\hat{F}=0$ for $\omega=0$.)

A suitable assumption for the dissipative term P_2 is the following eigenvalue condition.

Assumption 2.1. *The eigenvalues of* $P_2(\mathrm{i}\omega')$ *are non-positive.*

This assumption ensures that the initial value problem is well posed. In our example (1.2) we have

$$P_2(\mathrm{i}\omega')=\begin{pmatrix}-|\omega'|^2 & 0 & 0\\ 0 & -|\omega'|^2 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Thus the above eigenvalue condition is satisfied. The next assumption concerns the large part $\epsilon^{-1} P_0$ in our system.

Assumption 2.2. *For each* ω', *the eigenvalues* $\kappa = \kappa(\omega')$ *of* $P_0(\mathrm{i}\omega')$ *split into two groups* $M_{1,2} = M_{1,2}(\omega')$ *in the following way:*
If $\kappa \in M_1$, *then* $|\kappa| \geqslant 1$. *If* $\kappa \in M_2$, *then* $\kappa = 0$.

For our (symmetrized) example (1.2) we obtain

$$P_0(\mathrm{i}\omega') = -\begin{pmatrix} 0 & 0 & \mathrm{i}\omega_1' \\ 0 & 0 & \mathrm{i}\omega_2' \\ \mathrm{i}\omega_1' & \mathrm{i}\omega_2' & 0 \end{pmatrix}, \tag{2.3}$$

i.e.

$$\kappa_{1,2} = \pm \mathrm{i}\,|\omega'|, \quad \kappa_3 = 0.$$

Thus M_1 consists of two and M_2 of one eigenvalue.

In the general case, Assumption 2.2 ensures existence of a unitary transformation $U_0(\omega')$ such that

$$U_0^*(\omega')\,P_0(\mathrm{i}\omega')\,U_0(\omega') = \begin{pmatrix} R_0(\omega') & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.4}$$

where the eigenvalues of $R_0(\omega')$ are exactly the values in M_1. Therefore,

$$|R_0^{-1}(\omega')| \leqslant 1. \tag{2.5}$$

Now consider the matrix $\hat{P}(\omega)$ given in (2.2). If $\epsilon\nu\,|\omega| \ll 1$ we can determine a transformation

$$S = U_0(\omega')\,(I + \epsilon U_1(\omega'))\,(I + \nu\epsilon\,|\omega|\,T(\omega', |\omega|)),$$

which transforms $\hat{P}(\omega)$ to blockdiagonal form, i.e.

$$S^{-1}\hat{P}(\omega)\,S = \begin{pmatrix} \epsilon^{-1}R_0 + R_{11} + \nu\,|\omega|\,Q_{11} & 0 \\ 0 & R_{22} + \nu\,|\omega|\,Q_{22} \end{pmatrix}. \tag{2.6}$$

(Note that $0 < \epsilon \ll 1$ is always assumed.) Thus, for $\epsilon\nu\,|\omega| \ll 1$, we can introduce new variables $\hat{v}(\omega, t)$ into (2.1) by

$$\hat{u}(\omega, t) = S\hat{v}(\omega, t)$$

and obtain

$$\left.\begin{aligned} \hat{v}_t^{\mathrm{I}} &= |\omega|\,(\epsilon^{-1}R_0 + R_{11} + \nu\,|\omega|\,Q_{11})\,\hat{v}^{\mathrm{I}} + (S^{-1}\hat{F})^{\mathrm{I}}, \\ \hat{v}_t^{\mathrm{II}} &= |\omega|\,(R_{22} + \nu\,|\omega|\,Q_{22})\,\hat{v}^{\mathrm{II}} + (S^{-1}\hat{F})^{\mathrm{II}}, \\ \hat{v}(\omega, 0) &= S^{-1}\hat{f}. \end{aligned}\right\} \tag{2.7}$$

In this case, $\hat{v}^{\mathrm{I}}$ is highly oscillatory whereas $\hat{v}^{\mathrm{II}}$ varies slowly. We can decompose the whole solution u correspondingly if we make:

Assumption 2.3. *The initial data f and the forcing function F satisfy*

$$\hat{f}(\omega) = \hat{F}(\omega, t) \equiv 0 \quad \text{for} \quad |\omega| \geqslant \delta(\nu\epsilon)^{-1}, \quad 0 < \delta = \text{const.} \ll 1. \tag{2.8}$$

Under these assumptions the solution can be decomposed as

$$u = u^{\mathrm{I}} + u^{\mathrm{II}}, \tag{2.9}$$

where u^{I} varies on the 'fast' and u^{II} on the 'slow' scale.

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The assumption (2.8) is not a strong restriction in applications. For example, in low-Mach-number flows one has

$$|\hat{u}(\omega,t)| \approx \mathrm{e}^{-\gamma\sqrt{\nu|\omega|}}, \quad \gamma > 0.$$

Therefore, if $|\omega| \geqslant \delta(\nu\epsilon)^{-1}$, then $\hat{u}(\omega,t)$ is exponentially small for $0 < \gamma\delta/\epsilon\sqrt{\nu} \ll 1$.

3. Nonlinear systems

In this section we consider the nonlinear system (1.1). We could use the theory of pseudo-differential operators to *block diagonalize* the operator. However, we will proceed in a more elementary manner. As a first step, we estimate all space derivatives of the solution u independently of ϵ^{-1}.

Theorem 3.1. *Consider the system* (1.1) *under the Assumption* 2.1. *There is a time interval* $0 \leqslant t \leqslant T$, $T > 0$, *and a constant* K, *both independent of* ϵ *but depending on* $\|u(\cdot,0)\|_p$ *and* $\max_{0\leqslant\xi\leqslant T}\|F(\cdot,\xi)\|_p$, *such that*

$$\|u(\cdot,t)\|_p \leqslant K(\|u(\cdot,0)\|_p + \max_{0\leqslant\xi\leqslant t}\|F(\cdot,\xi)\|_p), \quad p \geqslant [\tfrac{1}{2}s]+2, \tag{3.1}$$

for $0 \leqslant t \leqslant T$. *Here*

$$\|u(\cdot,t)\|_p^2 = \sum_{|j|\leqslant p}\|D_1^{j_1}\dots D_s^{j_s}u(\cdot,t)\|^2, \quad |j| = \sum_i j_i, \quad D_i = \frac{\partial}{\partial x_i},$$

and $\|\cdot\|$ *denotes the* L_2*-norm*,

$$\|v\|^2 = \int_0^{2\pi}\dots\int_0^{2\pi}\sum_{l=1}^{n}|v^{(l)}|^2\,\mathrm{d}x_1\dots\mathrm{d}x_s.$$

We shall only sketch the proof. For more details, see Browning & Kreiss (1982) and Kreiss & Lorenz (1989).

Proof. We first consider (1.1) without the $\epsilon^{-1}P_0u$-term, i.e. we consider the system

$$w_t = (P_1+\nu P_2)\,w+F. \tag{3.2}$$

Applying $D^j = D_1^{j_1}\dots D_s^{j_s}$ to (3.2) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|D^jw\|^2 = 2(D^jw, D^jw_t)$$

$$= 2(D^jw, D^jP_1w)+2\nu(D^jw, P_2, D^jw)+2(D^jw, D^jF). \tag{3.3}$$

By Assumption 2.1 we have

$$(D^jw, P_2D^jw) \leqslant 0.$$

Since $(D^jw, P_1D^jw) = 0$, integration by parts and Sobolev inequalities give us

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(\cdot,t)\|_p^2 \leqslant H(\|w(\cdot,t)\|_p^2)+\|F(\cdot,t)\|_p^2,$$

where $H(\rho)$ is a polynomial in ρ. Therefore, the estimate (3.1) follows for the system (3.2).

Now consider the system (1.1) including the large term $\epsilon^{-1}P_0u$. For $(\mathrm{d}/\mathrm{d}t)\,\|D^ju\|^2$ we obtain again the relation (3.3) because

$$(D^ju, P_0D^ju) = 0.$$

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Therefore, the same estimates hold, and the theorem follows.

Theorem 3.1 implies that the solution of (1.1) is smooth in space, with bounds for the space derivatives independent of ϵ. Therefore, we can expand the solution of (1.1) into a Fourier series

$$u(x,t) = \sum_{\omega \in \mathbb{Z}^s} \hat{u}(\omega, t)\, \mathrm{e}^{\mathrm{i}\langle \omega, x\rangle},$$

and the series converges rapidly. (More precisely, truncation errors can be bounded independently of ϵ.) In general, this is true only in the time interval $0 \leqslant t \leqslant T$ specified in the theorem. However, in applications the dissipation operator often ensures existence and smoothness of the solution for all time. Then the convergence rate of the Fourier series will depend on ν.

If one is not interested in the fast scale of the problem, one initializes the data using the

Bounded derivative principle. *Choose the initial data such that $p \geqslant 1$ time derivatives are bounded independently of ϵ at $t = 0$.*

This is justified since one can show that if p time derivatives are bounded independently of ϵ at $t = 0$, then the same is true at later times. To make the results more precise, we define

Definition 3.2. *Let $T > 0$ and let $w(x, t, \epsilon)$ denote a vector function defined for $x \in \mathbb{R}^s$, $0 \leqslant t \leqslant T$, $0 < \epsilon \leqslant \epsilon_0$. We assume that w is 2π-periodic in each x_j. We say that w is slow to order p in $0 \leqslant t \leqslant T$ if all space derivatives $D^k w$ have p continuous time derivatives in $0 \leqslant t \leqslant T$ and*

$$\sup_{0 < \epsilon \leqslant \epsilon_0} \max_{0 \leqslant t \leqslant T} \left\| \frac{\partial^j D^k w}{\partial t^j}(\cdot, t, \epsilon) \right\| < \infty \tag{3.4}$$

for $j = 1, 2, \ldots, p$ and all space derivatives $D^k w$.

We say that w is slow to any order in $0 \leqslant t \leqslant T$ if w is slow to order p in $0 \leqslant t \leqslant T$ for any p. Correspondingly, if $T = \infty$, then condition (3.4) *becomes*

$$\sup_{0 < \epsilon \leqslant \epsilon_0} \sup_{0 \leqslant t < \infty} \left\| \frac{\partial^j D^k w}{\partial t^j}(\cdot, t, \epsilon) \right\| < \infty. \tag{3.5}$$

We can prove

Theorem 3.2. *If the initial data of* (1.1) *are chosen such that p time derivatives are bounded independently of ϵ at $t = 0$, then the solution is slow to order p in some time interval $0 \leqslant t \leqslant T$. Here T does not depend on ϵ.*

Proof. As before, we can use integration by parts and Sobolev inequalities to estimate both space and time derivatives. This proves the theorem. For more details, see Browning & Kreiss (1982).

We now use Assumption 2.2. If M_1 contains r eigenvalues, then the bounded derivative principle defines r relations $\hat{v}^{\mathrm{I}}(\omega, 0) = 0$, which the initial data must satisfy. We shall derive these relations in an asymptotic sense. To this end, let $U_0(\omega')$ denote the transformation (2.4) and let $h = h(x)$ be an arbitrary L_2-function (with values in $\mathbb{R}^n$),

$$h(x) = \sum_{\omega} \mathrm{e}^{\mathrm{i}\langle \omega, x\rangle} \hat{h}(\omega).$$

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We define the projection operator Q by

$$\left.\begin{aligned} h^{\mathrm{I}} &= Qh = \sum_{\omega \neq 0} U_0(\omega') \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} U_0^*(\omega')\, \mathrm{e}^{\mathrm{i}\langle \omega, x\rangle}\, \hat{h}(\omega), \\ h^{\mathrm{II}} &= (I-Q)\,h = \hat{h}(0) + \sum_{\omega \neq 0} U_0(\omega') \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} U_0^*(\omega')\, \mathrm{e}^{\mathrm{i}\langle \omega, x\rangle} \hat{h}(\omega). \end{aligned}\right\} \tag{3.6}$$

(The decomposition $h = h^{\mathrm{I}} + h^{\mathrm{II}}$ constructed here is much easier to determine than the decomposition (2.9) because only the matrix $U_0(\omega')$ is involved here, in contrast to the matrix S of the transformation (2.6).) In the expression

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

the unit matrix I has the dimension $r \times r$, i.e. I has the same dimension as $\hat{R}_0$. We set $L_2^{\mathrm{I}} = QL_2, L_2^{\mathrm{II}} = (I-Q)\,L_2$ and obtain the orthogonal decomposition $L_2 = L_2^{\mathrm{I}} \oplus L_2^{\mathrm{II}}$. On L_2^{I} the operator P_0 is elliptic and has a bounded inverse. (Recall the estimate (2.5).)

Using the projection operator Q, we rewrite the system (1.1) in the form

$$u_t^{\mathrm{I}} = \epsilon^{-1} P_0\, u^{\mathrm{I}} + (P_1(u, \partial/\partial x)\, u)^{\mathrm{I}} + \nu (P_2(\partial/\partial x)\, u)^{\mathrm{I}} + F^{\mathrm{I}}, \tag{3.7a}$$

$$u_t^{\mathrm{II}} = (P_1(u, \partial/\partial x)\, u)^{\mathrm{II}} + \nu (P_2(\partial/\partial x)\, u)^{\mathrm{II}} + F^{\mathrm{II}}, \quad u = u^{\mathrm{I}} + u^{\mathrm{II}}. \tag{3.7b}$$

(Here $F^{\mathrm{I}} = QF$, etc.) We will show that if u is slow to order p then u^{I} is determined by u^{II} up to terms of order $O(\epsilon^p)$. The first time derivative $\partial u/\partial t$ is bounded independently of ϵ if and only if $P_0\, u^{\mathrm{I}} = O(\epsilon)$, or, equivalently,

$$u^{\mathrm{I}} = \epsilon u_1^{\mathrm{I}}, \quad \text{where} \quad u_1^{\mathrm{I}} = O(1). \tag{3.8}$$

(For a C^∞-function $u = u(x, \epsilon)$ we write $u = O(\epsilon)$ if the L_2-norm of each space derivative of u is $O(\epsilon)$. For projections like $P_0\, u^{\mathrm{I}}$, space derivatives can be defined using the Fourier expansion; then the $O(\epsilon)$-terminology is used analogously.) Condition (3.8) requires that u has, to first approximation, no component in L_2^{I}. Therefore, if u is slow to order $p \geqslant 1$, then to first approximation the solution of (3.7) is determined by

$$u^{\mathrm{I}} \equiv 0, \quad u_t^{\mathrm{II}} = (P_1(u^{\mathrm{II}}, \partial/\partial x)\, u^{\mathrm{II}})^{\mathrm{II}} + \nu (P_2(\partial/\partial x)\, u^{\mathrm{II}})^{\mathrm{II}} + F^{\mathrm{II}}.$$

Now let us assume (3.8). Differentiation of (3.7) with respect to t yields

$$\begin{aligned} u_{tt}^{\mathrm{I}} &= \epsilon^{-1} P_0\, u^{\mathrm{I}} t + O(1) \\ &= \epsilon^{-1} P_0 (\epsilon^{-1} P_0\, u_1^{\mathrm{I}} + (P_1(u^{\mathrm{II}}, \partial/\partial x)\, u^{\mathrm{II}})^{\mathrm{I}} + \nu (P_2(\partial/\partial x)\, u^{\mathrm{II}})^{\mathrm{I}} + F^{\mathrm{I}}) + O(1), \\ u_{tt}^{\mathrm{II}} &= O(1). \end{aligned}$$

Therefore, the second time derivative of u is bounded independently of ϵ if and only if

$$\epsilon^{-1} P_0\, u_1^{\mathrm{I}} + (P_1(u^{\mathrm{II}}, \partial/\partial x)\, u^{\mathrm{II}})^{\mathrm{I}} + \nu (P_2(\partial/\partial x)\, u^{\mathrm{II}})^{\mathrm{I}} + F^{\mathrm{I}} = O(\epsilon).$$

This equation determines u^{I} by u^{II} up to terms of order $O(\epsilon^2)$. This process can be continued and we can obtain the desired relation between u^{I} and u^{II} to any order in ϵ.

To summarize, the bounded derivative principle determines for any $p \geqslant 1$ a 'slow manifold' $\mathscr{M}_p$ up to terms of order $O(\epsilon^p)$. Initializing p time derivatives means the same as choosing initial data on $\mathscr{M}_p$. For these initial data, the solution will be slow to order p in any time interval where we can estimate the solution as in Theorem 3.1. In this time interval, $u^{\mathrm{I}}(\cdot,t)$ is determined by $u^{\mathrm{II}}(\cdot,t)$ except for terms of order $O(\epsilon^p)$. In general, the manifolds $\mathscr{M}_p$ are not invariant manifolds, however; to construct slow invariant manifolds (in the sense of dynamical systems) is a much more difficult task. In the next section we will discuss this question by replacing the partial differential system by a finite system of ordinary differential equations.

4. Reduction to systems of ordinary differential equations

In §3 we have shown that for smooth data the solution of (1.1) can be expanded into a rapidly convergent Fourier series

$$u(x,t) = \sum_{\omega \in \mathbb{Z}^s} \mathrm{e}^{\mathrm{i}\langle \omega, x\rangle}\, \hat{u}(\omega,t). \tag{4.1}$$

Since the space derivatives of u obey bounds with constants independent of ϵ, it is reasonable to assume that the solution can be well represented by a truncated Fourier series,

$$\sum_{|\omega| \leqslant N} \mathrm{e}^{\mathrm{i}\langle \omega, x\rangle}\, \hat{u}(\omega,t), \tag{4.2}$$

where $N\epsilon \ll 1$. Also, since we are only interested in real solutions we assume that

$$\hat{u}(\omega,t) = \overline{\hat{u}(-\omega,t)}.$$

Introducing (4.2) into (3.7) and neglecting all terms with wave vectors $|\omega| > N$, we obtain a real system of ordinary differential equations

$$y' = \epsilon^{-1}Ay + C(v,y,t)\,y + f(v,t), \quad v' = g(v,y,t), \quad y(0) = y_0, \quad v(0) = v_0. \tag{4.3}$$

Here the components of y and v consist of

$$u^{\mathrm{I}}(\omega,t) + u^{\mathrm{I}}(-\omega,t), \quad \mathrm{i}(u^{\mathrm{I}}(\omega,t) - u^{\mathrm{I}}(-\omega,t))$$

and

$$u^{\mathrm{II}}(\omega,t) + u^{\mathrm{II}}(-\omega,t), \quad \mathrm{i}(u^{\mathrm{II}}(\omega,t) - u^{\mathrm{II}}(-\omega,t)),$$

respectively. The constant matrix $A = -A^*$ is non-singular and anti-hermitean. The functions C, f and g are C^∞-smooth with respect to all variables. We are only interested in a bounded solution (y,v) of (4.3). Therefore, we can (and will) assume that the functions C, f and g vanish identically for sufficiently large $|y|$, $|v|$. In applications, one can always establish an *a priori* energy estimate for (1.1), which carries over to the truncated system (4.3) and therefore the above assumption is not restrictive.

All our discussions in §3 carry over to systems of the form (4.3). We refer to Kreiss & Lorenz (1994). Here we will only discuss the existence of a slow invariant manifold in the sense of dynamical systems. (Even if the system (4.3) is autonomous, the constructed manifold will depend on time.)

If $f(v,t) \equiv 0$ then the existence of a slow invariant manifold is clear. It is defined by

$$y \equiv 0.$$

This motivates to construct a substitution

$$y = \Phi(v,t,\epsilon) + \tilde{y} \tag{4.4}$$

such that $\tilde{f}(v,t) \equiv 0$. Unfortunately, in general, we can determine such a substitution only in a finite time interval $0 \leqslant t \leqslant T$. If this is the case, we only obtain the existence of a local slow manifold. Under rather precise additional assumptions we can prove the existence for all times.

Introducing (4.4) into (4.3) and observing that

$$\begin{aligned} \epsilon y_t &= \epsilon \tilde{y}_t + \epsilon \Phi_v v_t + \epsilon \Phi_t \\ &= \epsilon \tilde{y}_t + \epsilon \Phi_v g(v, \Phi + \tilde{y}, t) + \epsilon \Phi_t, \end{aligned}$$

we obtain for $\tilde{y}$

$$\epsilon \tilde{y}_t = (A + \epsilon \tilde{C}(v, \tilde{y}, t))\,\tilde{y} + \tilde{F},$$

where $$-\tilde{F} = \epsilon(\Phi_t + \Phi_v g(v, \Phi, t)) - (A + \epsilon C(v, \Phi, t))\,\Phi - f(v,t).$$

We want to choose Φ in such a way that

$$\tilde{F} \equiv 0,$$

i.e. we want to choose Φ as a solution of the system of partial differential equations

$$\Phi_t + \Phi_v g(v, \Phi, t) = (\epsilon^{-1} A + C(v, \Phi, t))\,\Phi + f(v,t). \tag{4.5}$$

Note that

$$\Phi_v g(v, \Phi, t) = \sum_{\nu=1}^{M} g^{(\nu)}(v, \Phi, t)\, D_\nu \Phi, \quad D_\nu = \frac{\partial}{\partial v_\nu}. \tag{4.6}$$

(Here we denote the components of g by $g^{(1)}, \ldots, g^{(M)}$. The number M is the number of slow variables in (4.3), or, equivalently, the number of slow equations.) Thus, (4.5) is a quasi-linear hyperbolic system for Φ, and if Φ has m components then the principle part of (4.5) consists of m scalar expressions for the components of Φ; coupling occurs through zero-order terms only.

We construct a solution of (4.5) in the following way. Let $\alpha(t)$ be a monotone C^∞-cut-off function with

$$\alpha(t) = \begin{cases} 0 & \text{for} \quad -1 \leqslant t \leqslant -\tfrac{2}{3}, \\ 1 & \text{for} \quad -\tfrac{1}{3} \leqslant t < \infty. \end{cases}$$

We replace (4.5) by

$$\Phi_t + \Phi_v g(v, \Phi, t) = (\epsilon^{-1} A + C(v, \Phi, t))\,\Phi + \alpha(t) f(v,t), \quad t \geqslant -1, \tag{4.7a}$$

and solve the equation with initial data

$$\Phi(v, -1) = 0. \tag{4.7b}$$

To reduce the size of the inhomogeneous term in (4.7a), we write

$$\Phi(v,t) = \epsilon \phi_0(v,t) + \Phi_1(v,t), \quad \phi_0 = -\alpha(t) A^{-1} f(v,t).$$

For Φ_1 we obtain a system of the same type

$$\begin{gathered} \Phi_{1t} + \Phi_{1v} g(v, \epsilon\phi_0 + \Phi_1, t) = (\epsilon^{-1} A + C_1(v, \Phi_1, t))\,\Phi_1 + \epsilon \alpha(t) f_1(v,t), \\ \Phi_1(v, -1) = 0, \end{gathered}$$

with a forcing of order $O(\epsilon)$. The reduction of the inhomogeneous term can be continued. One can construct uniformly smooth functions $\phi_0, \phi_1, \ldots,$ with the following property. If one sets

$$\Phi = \epsilon\phi_0 + \epsilon^2 \phi_1 + \ldots + \epsilon^q \phi_{q-1} + \Phi_q =: \psi_q + \Phi_q,$$

then Φ solves (4.7) if and only if Φ_q solves the initial value problem

$$\left.\begin{aligned} \Phi_{qt}+\Phi_{qv}\,g(v,\psi_q+\Phi_q,t) &= (\epsilon^{-1}A+C_q(v,\Phi_q,t))\,\Phi_q+\epsilon^q\alpha(t)f_q(v,t),\\ \Phi_q(v,-1) &= 0. \end{aligned}\right\} \tag{4.8}$$

Let $$\Phi_q = \epsilon^q \Psi.$$

For Ψ we obtain

$$\left.\begin{aligned} \Psi_t+\sum_{\nu=1}^{n} a_\nu(v,\epsilon^q\Psi,t)\,D_\nu\,\Psi &= (\epsilon^{-1}A+B(v,\epsilon^q\Psi,t))\,\Psi+b(v,t),\\ \Psi(v,-1) &= 0, \end{aligned}\right\} \tag{4.9}$$

with

$$a_\nu(v,y,t) = g^{(\nu)}(v,\psi_q(v,t)+y,t),\quad B(v,y,t) = C_q(v,y,t),\quad b(v,t) = \alpha(t)f_q(v,t).$$

In (4.9) the nonlinearities are weak for small ϵ. Therefore we can use standard energy estimates to bound all v-derivatives of Ψ independently of $0<\epsilon\leqslant\epsilon_0$ in any finite time interval $-1\leqslant t\leqslant T$, provided $\epsilon_0=\epsilon_0(T)$ is sufficiently small. Neither T nor ϵ_0 depend on the order of the v-derivative.

Now we can use as initial data $\tilde{y}(0)\equiv 0$, or in the original variables

$$y(0) = \Phi(v(0),0,\epsilon); \tag{4.10}$$

then $\tilde{y}\equiv 0$ for $0\leqslant t\leqslant T$, or in the original variables,

$$y(t) = \Phi(v(t),t,\epsilon),\quad 0\leqslant t\leqslant T. \tag{4.11}$$

We express our result in terms of existence of a slow manifold.

Definition 4.2. *The system* (4.3) *has a local slow manifold if for any* $T>0$ *there is an* $\epsilon_0(T)>0$ *such that* (4.5) *has a uniformly smooth solution* $\Phi=\Phi(v,t,\epsilon)$ *defined for all* v, $0\leqslant t\leqslant T$, $0<\epsilon\leqslant\epsilon_0(T)$. *The system* (4.3) *has a global slow manifold, if there is an* $\epsilon_0>0$ *such that* (4.7) *has a uniformly smooth solution* $\Phi=\Phi(v,t,\epsilon)$ *defined for all* v, $0\leqslant t<\infty$, $0<\epsilon\leqslant\epsilon_0$.

We have shown

Theorem 4.1. *The system* (4.3) *has a local slow manifold.*

The manifold exists as long as the solution of (4.7) does not generate any shocks. Shocks occur if characteristics starting from different points at $t=-1$ cross each other. Therefore, the most precise estimates can be obtained by estimating the solution and its derivatives using the method of characteristics. The characteristics of (4.7) are defined by

$$\mathrm{d}v(t)/\mathrm{d}t = g(v,\Phi(v,t),t),\quad v(-1)=v_0. \tag{4.12}$$

Along the characteristics, Φ is the solutions of

$$\mathrm{d}\Phi/\mathrm{d}t = (\epsilon^{-1}A+C(v,\Phi,t))\,\Phi+\alpha(t)f(v,t),\quad \Phi(v_0,-1)=0. \tag{4.13}$$

We know that (4.7) has a solution in some time interval. Therefore, we can think of $g(v,\Phi(v,t),t)=:\tilde{g}(v,t)$ and $C(v,\Phi(v,t),\,t)=:\tilde{C}(v,t)$ as given functions of v,t.

The Appendix tells us that the convergence or divergence of the characteristics is governed by the solution operator $S_V=S_V(t,t_0,v_0)$ of the linearized system (4.12),

$$\mathrm{d}V/\mathrm{d}t = \tilde{g}_v\,V.$$

Denoting by $S_\Phi = S_\Phi(t, t_0, v_0)$ the solution operator of

$$\mathrm{d}\Phi/\mathrm{d}t = (\epsilon^{-1}A + \tilde{C})\,\Phi,$$

the Appendix gives us

Theorem 4.2. *The first p derivatives of Φ with respect to v can be estimated for $0 \leqslant t \leqslant T$ in terms of*

$$\left.\begin{array}{l}\sup\{|S_\Phi(t,-1,v_0)|\,|S_V^{-1}(t,-1,v_0)|^j : 0 \leqslant t \leqslant T,\quad v_0 \in \mathbb{R}^M\} \quad \textit{and} \\ \sup\left\{\int_{-1}^{t} |S_\Phi(t,\xi,v_0)|\,|S_V^{-1}(t,\xi,v_0)|^j\,\mathrm{d}\xi : 0 \leqslant t \leqslant T,\quad v_0 \in \mathbb{R}^M\right\},\quad j = 0,\ldots,p.\end{array}\right\} \tag{4.14}$$

If these expressions stay uniformly bounded for all T, then there is a global manifold.

In applications, due to scattering, the fast waves die out much more rapidly than the perturbations in v decay. Therefore, the conditions (4.14) appear to be reasonable. Also, these conditions are exactly those encountered when studying the existence of invariant manifold under perturbations (see, for example, Fenichel 1971).

Appendix A. Estimates for solutions of hyperbolic systems with scalar principle parts

We consider systems of the form

$$\left.\begin{array}{c}\dfrac{\partial u}{\partial t} + \sum_{j=1}^{s} a_j(x,t)\,D_j u = B(x,t)\,u + F(x,t), \quad D_j = \dfrac{\partial}{\partial x_j}, \\ u(x,0) = f(x).\end{array}\right\} \tag{A 1}$$

Here we use the notations $x = (x_1, \ldots, x_s)$, $u = (u^{(1)}, \ldots, u^{(n)})'$, $a = (a_1, \ldots, a_s)$. The a_j are scalars and B is an $n \times n$ matrix; all coefficients and data are real, C^∞-smooth, and bounded, for simplicity.

We can solve (A 1) by the method of characteristics. The characteristics are the solutions of the ordinary differential equations

$$\mathrm{d}x(t)/\mathrm{d}t = a(x(t), t), \quad x(0) = x_0. \tag{A 2}$$

Along every characteristic the solution of (A 1) is determined by

$$\mathrm{d}u(x(t),t)/\mathrm{d}t = B(x(t),t)\,u + F(x(t),t), \quad u(x(0),0) = f(x(0)). \tag{A 3}$$

Denoting by $S_B(t, \xi, x_0)$ the solution operator of (A 3) we can write the solution of (A 3) in the form

$$u((x(t),t) = S_B(t,0,x_0)f(x_0) + \int_0^t S_B(t,\xi,x_0)\,F(x(\xi),\xi)\,\mathrm{d}\xi. \tag{A 4}$$

Therefore we obtain

Lemma A 1. *Assume that*

$$\sup_{t,x_0} |S_B(t,0,x_0)| \leqslant K_{10}, \sup_{t,x_0} \int_0^t |S_B(t,\xi,x_0)|\,\mathrm{d}\xi \leqslant K_{11}; \tag{A 5a}$$

then

$$|u|_\infty \leqslant K_{10}\,|f|_\infty + K_{11}\,|F|_\infty. \tag{A 5b}$$

We will now estimate the first derivatives of u. The vector of first derivatives $Du = (D_1 u, \ldots, D_s u)'$ solves

$$\frac{\partial}{\partial t} Du + \sum_{j=1}^{s} a_j(x,t) D_j(Du) = (A_1 + B_1) Du + F_1, \qquad Du(x,0) = Df(x). \tag{A 6}$$

Here

$$A_1 = -\begin{pmatrix} (D_1 a_1) I & \ldots & (D_1 a_s) I \\ (D_2 a_1) I & \ldots & (D_2 a_s) I \\ \cdots & \cdots & \cdots \\ (D_s a_1) I & \ldots & (D_s a_s) I \end{pmatrix}, \quad I = \begin{pmatrix} 1 & \ldots & 0 \\ & \ddots & \\ 0 & \ldots & 1 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} B & \ldots & 0 \\ & \ddots & \\ 0 & \ldots & B \end{pmatrix}, \quad F_1 = \begin{pmatrix} D_1 F + (D_1 B) u \\ \vdots \\ D_s F + (D_s B) u \end{pmatrix}.$$

We need

Lemma A 2. *For any x, t, $\bar{x}$, $\bar{t}$ it holds that*

$$A_1(x,t) B_1(\bar{x}, \bar{t}) = B_1(\bar{x}, \bar{t}) A_1(x,t). \tag{A 7}$$

Furthermore, there exists a permutation matrix $P = P^{\mathrm{T}}$ such that

$$-PA_1 P = \begin{pmatrix} A & \ldots & 0 \\ & \ddots & \\ 0 & \ldots & A \end{pmatrix}, \quad A = \begin{pmatrix} D_1 a_1 & \ldots & D_1 a_s \\ D_2 a_1 & \ldots & D_2 a_s \\ \cdots & \cdots & \cdots \\ D_s a_1 & \ldots & D_s a_s \end{pmatrix}. \tag{A 8}$$

Proof. Equation (A 7) follows directly from the definition of A_1, B_1. To show (A 8) we write (A 6) in another order: Setting $Du^{(j)} = (D_1 u^{(j)}, \ldots, D_s u^{(j)})'$ and

$$\tilde{D}u = (Du^{(1)}, \ldots, Du^{(n)})',$$

we rewrite (A 6) in terms of $\tilde{D}u$ to obtain (A 8).

As before, we can solve (A 6) by the method of characteristics. Along the characteristics Du is the solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}(Du) = (A_1 + B_1) Du + F_1. \tag{A 9}$$

By the preceding lemma, the solution operator of (A 9) can be written in the form

$$S(t, \xi) = S_1(t, \xi) S_2(t, \xi),$$

where S_1, S_2 are the solution operators of

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = -P \begin{pmatrix} A & \ldots & 0 \\ & \ddots & \\ 0 & \ldots & A \end{pmatrix} P u_1, \tag{A 10}$$

and

$$\frac{\mathrm{d}u_2}{\mathrm{d}t} = \begin{pmatrix} B & \ldots & 0 \\ & \ddots & \\ 0 & \ldots & B \end{pmatrix} u_2, \tag{A 11}$$

respectively. Therefore,

$$S_1(t,\xi) = P\begin{pmatrix} S_{-A}(t,\xi) & \dots & 0 \\ & \ddots & \\ 0 & \dots & S_{-A}(t,\xi)\end{pmatrix} P, \tag{A 12}$$

$$S_2(t,\xi) = \begin{pmatrix} S_{B}(t,\xi) & \dots & 0 \\ & \ddots & \\ 0 & \dots & S_{B}(t,\xi)\end{pmatrix}, \tag{A 13}$$

where S_A and S_B are the solution operators of

$$\mathrm{d}v/\mathrm{d}t = -Av \quad \text{and} \quad \mathrm{d}w/\mathrm{d}t = Bw,$$

respectively. We have proved

Theorem A 1. *If* $|S_B(t,\xi,x_0)|$ *and* $|S_{-A}(t,\xi,x_0)|\,|S_B(t,\xi,x_0)|$ *satisfy the estimate* (A 5*a*), *then we have*

$$|Du|_\infty \leqslant \text{const.}\,(|Df|_\infty + |DF|_\infty + |F|_\infty).$$

We can now repeat the process. We derive an equation for $D^2u = (D_1Du, \dots, D_sDu)'$. By the same argument as before we can write the solution operator of the ordinary differential equations along the characteristics in the form

$$S(t,\xi) = S_{11}(t,\xi)\,S_{12}(t,\xi)\,S_2(t,\xi),$$

where S_{11}, S_{12} and S_2 are of the form (A 12) and (A 13), respectively.

By induction we obtain

Theorem A 2. *If* $|S_B(t,\xi,x_0)|$ *and* $|S_{-A}(t,\xi,x_0)|^p\,|S_B(t,\xi,x_0)|$ *satisfy the estimate* (A 5*a*), *then we have*

$$|D^pu|_\infty \leqslant \text{const.}\left(\sum_{j=0}^{p} (|D^jf|_\infty + |D^jF|_\infty)\right).$$

We will now derive a geometric interpretation of the estimate for S_{-A}. The equations for the characteristics are given by

$$\mathrm{d}x(t)/\mathrm{d}t = a(x(t),t), \quad x(\xi) = x_0. \tag{A 14}$$

(For later purposes we choose ξ as initial time.) Now consider a perturbation of the initial data

$$\mathrm{d}y(t)/\mathrm{d}t = a(y(t),t), \quad y(\xi) = x_0 + \delta v_0.$$

To first approximation, the difference $v(t) = y(t) - x(t)$ satisfies the linearized differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = A^*v, \quad A^* = \begin{pmatrix} D_1a_1 & \dots & D_sa_1 \\ D_1a_2 & \dots & D_sa_2 \\ \dots & \dots & \dots \\ D_1a_s & \dots & D_sa_s \end{pmatrix}, \tag{A 15}$$

i.e.

$$|S_{A^*}^{-1}(t,\xi)\,(y(t)-x(t))| = |y(\xi)-x(\xi)|.$$

Thus $S_{A^*}^{-1}(t,\xi)$ measures how much the distance of the characteristics change with time. If the characteristics converge or diverge with increasing time, then $|S_{A^*}^{-1}|$

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becomes larger or smaller with increasing time, respectively. The following lemma shows that

$$|S_{A^*}^{-1}| = |S_{-A}|. \tag{A 16}$$

Lemma A 4. *Consider the systems*

$$\mathrm{d}v/\mathrm{d}t = A^*v, \quad \mathrm{d}w/\mathrm{d}t = -Aw.$$

For the solution operators we have

$$(S_{A^*}^{-1})^* = S_{-A}.$$

Proof. We approximate the first system by the explicit Euler scheme

$$v(t+h) = (I+hA^*(t))\,v(t).$$

For the discrete solution operator $S_{A^*}^h$ we have

$$S_{A^*}^h(t,\xi) = (I+hA^*(t-h))\dots(I+hA^*(\xi)),$$

i.e.

$$\begin{aligned}((S_{A^*}^h(t,\xi))^{-1})^* &= (I+hA(t-h))^{-1}\dots(I+hA(\xi))^{-1}\\ &= (I-hA(t-h)+O(h^2))\dots(I-hA(\xi)+O(h^2)).\end{aligned}$$

The limit process $h\to 0$ gives us the desired relation. This proves the lemma.

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